National Exams May 2012

04-CHEM-B1, Transport Phenomena

3 hours duration

NOTES

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.

2. The examination is an OPEN BOOK EXAM.

3. Candidates may use any non-communicating calculator.

4. Not all problems are of equal weight.

5. Answer all four questions.

6. State all assumptions clearly.

7. The various conservation equations (momentum, energy, and continuity) are given in Tables 1-4 appended to this paper taken from Brodkey, R.S. and Hershey H.C. (1988) Transport Phenomena – A Unified Approach.
Q1. [20 marks] Consider the system shown in Fig. 1, in which a cylindrical rod is being moved axially in the z-direction with velocity $U$. The rod and the cylinder are co-axial. The inner radius of the cylinder is $R$ and the diameter of the rod $\kappa R$ such that $\kappa < 1$. Problems of this kind arise in the analysis of wire-coating dies such as those used in processing of thermoplastic materials.

![Diagram of fluid flow through a system with a rod and cylinder](image)

**Fig. 1: Annular flow with inner cylinder moving axially**

Starting with the appropriate form of the Navier-Stokes equations [see Table 1 on p 5], show that the steady-state velocity profile is given by:

$$u_z = U \frac{\ln(r/R)}{\ln(\kappa)}$$

Furthermore, show that the volumetric flow rate, $\dot{V}$, is given by:

$$\dot{V} = \frac{\pi R^2 U}{2} \left[ \frac{1}{\ln(\kappa)}(\kappa^2 - 1) - 2\kappa^2 \right]$$

Q2. [30 marks] A plane slab, of thickness $2L$ and thermal conductivity $k$, uniformly generates heat throughout at a rate $q_0$. The temperatures at the left and right faces are $T_1$ (at $x = -L$) and $T_2$ (at $x = +L$). Starting with the appropriate form of the energy equation [see Table 3 on p 7], show that the steady-state temperature profile throughout the slab is:

$$T(x) = T_1 + \frac{q_0}{2k} \left( L^2 - x^2 \right) + \frac{(T_2 - T_1)}{2} \left( 1 + \frac{x}{L} \right)$$

If both faces are in contact with air at $T_\infty$ and the convective heat transfer coefficient at the left face is $h_1$, and that at the right face is $h_2$, develop expressions for the surface temperatures $T_1$ and $T_2$. 

May 2012
Q3. [20 marks] An important unit operation is the absorption of one of the constituents of a gas mixture preferentially in a contacting liquid. In some cases, depending upon the nature of the molecules, the absorption may or may not involve simultaneous chemical reaction. Consider a liquid surface being exposed to a gas mixture that contains a component $A$, which preferentially dissolves in the liquid $B$ as shown in Fig. 2. At the liquid surface the concentration of $A$ is $C_{A_0}$. Throughout the liquid $A$ disappears according to a first-order chemical reaction with a specific reaction rate constant $k_1$. At some depth $\delta$ within the liquid the concentration of $A$ will have reached zero.

![Diagram](image)

**Fig. 2: Absorption of gas $A$ in liquid $b$ with chemical reaction**

Starting with the appropriate form of the species continuity equation [see Table 4 on p 7], show that the concentration profile of $A$ within the liquid $B$, is given by:

$$C_A = C_{A_0} \left\{ \cosh(mz) - \frac{\sinh(mz)}{\tanh(m\delta)} \right\}$$

in which $m = \sqrt{k_1/D_{AB}}$.

*Note:* The general solution to differential equations of the form $ay - b \frac{d^2y}{dx^2} = 0$ is given by

$$y = C_1 \cosh\left(\sqrt{a/b} \cdot x\right) + C_2 \sinh\left(\sqrt{a/b} \cdot x\right)$$

Q4. [30 marks] Figure 3 shows a liquid falling as a thin film under laminar flow down a vertical flat surface whilst being exposed to a gas $A$, which dissolves in the liquid. The liquid contains a uniform concentration of $A$ ($C_{A_0}$) at the top ($z = 0$), and at the liquid surface ($x = 0$) the concentration of gas is ($C_{A_1}$).
Fig. 3: Falling liquid film

Since this is a problem of fluid flow and mass transfer, first select the appropriate form of the Navier-Stokes equation [see Table 1 on p 5] and develop an expression for the applicable velocity distribution. Then select the appropriate form of the continuity equation for species A [see Table 4 on p 7], make the necessary simplifying assumptions and utilizing the resultant velocity distribution show that the governing differential equation for the problem is:

\[
\rho g \delta^2 \frac{1-(x/\delta)^2}{2\mu} \frac{\partial C_A}{\partial z} = D \frac{\partial^2 C_A}{\partial x^2}
\]

Define the boundary conditions but do not attempt to solve the equation.
Table 1: The Navier-Stokes equations for fluids of constant $\rho$ and $\mu$

Navier–Stokes equation in vector form (rectangular coordinates only)

$$\frac{\partial U}{\partial t} + (U \cdot \nabla) U = -\frac{1}{\rho} \nabla p + \nabla \left( \nu \nabla^2 U \right)$$ \hspace{1cm} (5.15)

Rectangular coordinates

\textbf{x component:} \hspace{1cm} \frac{\partial U_x}{\partial t} + U_x \frac{\partial U_x}{\partial x} + U_y \frac{\partial U_x}{\partial y} + U_z \frac{\partial U_x}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nabla \left( \nu \nabla^2 U \right) \hspace{1cm} \text{(A)}

\textbf{y component:} \hspace{1cm} \frac{\partial U_y}{\partial t} + U_x \frac{\partial U_y}{\partial x} + U_y \frac{\partial U_y}{\partial y} + U_z \frac{\partial U_y}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nabla \left( \nu \nabla^2 U \right) \hspace{1cm} \text{(B)}

\textbf{z component:} \hspace{1cm} \frac{\partial U_z}{\partial t} + U_x \frac{\partial U_z}{\partial x} + U_y \frac{\partial U_z}{\partial y} + U_z \frac{\partial U_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nabla \left( \nu \nabla^2 U \right) \hspace{1cm} \text{(C)}

Cylindrical coordinates

\textbf{r component:} \hspace{1cm} \frac{\partial U_r}{\partial t} + U_r \frac{\partial U_r}{\partial r} + \frac{U_x}{r} \frac{\partial U_r}{\partial \theta} + \frac{U_z}{r} \frac{\partial U_r}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial U_r}{\partial \theta} + \frac{\partial \left( \nu \partial^2 U_r \right)}{\partial \theta^2} \hspace{1cm} \text{(D)}

\textbf{\theta component:} \hspace{1cm} \frac{\partial U_\theta}{\partial t} + U_r \frac{\partial U_\theta}{\partial r} + \frac{U_x}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{U_z}{r} \frac{\partial U_\theta}{\partial z} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{\partial U_\theta}{\partial \theta} + \frac{\partial \left( \nu \partial^2 U_\theta \right)}{\partial \theta^2} \hspace{1cm} \text{(E)}

\textbf{z component:} \hspace{1cm} \frac{\partial U_z}{\partial t} + U_r \frac{\partial U_z}{\partial r} + \frac{U_x}{r} \frac{\partial U_z}{\partial \theta} + \frac{U_z}{r} \frac{\partial U_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial U_z}{\partial \theta} + \frac{\partial \left( \nu \partial^2 U_z \right)}{\partial \theta^2} \hspace{1cm} \text{(F)}

Spherical coordinates

\textbf{r component:} \hspace{1cm} \frac{\partial U_r}{\partial t} + U_r \frac{\partial U_r}{\partial r} + \frac{U_\theta}{r} \frac{\partial U_r}{\partial \theta} + \frac{U_\phi}{r \sin \theta} \frac{\partial U_r}{\partial \phi} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial U_r}{\partial \theta} + \frac{\partial \left( \nu \partial^2 U_r \right)}{\partial \theta^2} \hspace{1cm} \text{(G)}

\textbf{\theta component:} \hspace{1cm} \frac{\partial U_\theta}{\partial t} + U_r \frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{U_\phi}{r \sin \theta} \frac{\partial U_\theta}{\partial \phi} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{\partial U_\theta}{\partial \theta} + \frac{\partial \left( \nu \partial^2 U_\theta \right)}{\partial \theta^2} \hspace{1cm} \text{(H)}

\textbf{\phi component:} \hspace{1cm} \frac{\partial U_\phi}{\partial t} + U_r \frac{\partial U_\phi}{\partial r} + \frac{U_\theta}{r} \frac{\partial U_\phi}{\partial \theta} + \frac{U_\phi}{r \sin \theta} \frac{\partial U_\phi}{\partial \phi} = -\frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \phi} + \frac{\partial U_\phi}{\partial \phi} + \frac{\partial \left( \nu \partial^2 U_\phi \right)}{\partial \phi^2} \hspace{1cm} \text{(I)}

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1 Brodkey, R.S. and Hershey H.C. (1988) Transport Phenomena – A Unified Approach Table 5.7 p147.
Table 2: Shear stress-velocity gradient relationships for constant viscosity\(^2\)

<table>
<thead>
<tr>
<th>Rectangular coordinates</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_{xx} = -2\mu(\partial U_x / \partial x) + (2\mu/3)(\nabla \cdot U) )</td>
<td>(A)</td>
</tr>
<tr>
<td>( \tau_{yy} = -2\mu(\partial U_y / \partial y) + (2\mu/3)(\nabla \cdot U) )</td>
<td>(B)</td>
</tr>
<tr>
<td>( \tau_{zz} = -2\mu(\partial U_z / \partial z) + (2\mu/3)(\nabla \cdot U) )</td>
<td>(C)</td>
</tr>
<tr>
<td>( \tau_{xy} = \tau_{yx} = -\mu[(\partial U_y / \partial y) + (\partial U_x / \partial x)] )</td>
<td>(D)</td>
</tr>
<tr>
<td>( \tau_{xz} = \tau_{zx} = -\mu[(\partial U_x / \partial z) + (\partial U_z / \partial x)] )</td>
<td>(E)</td>
</tr>
<tr>
<td>( \tau_{yz} = \tau_{zy} = -\mu[(\partial U_y / \partial z) + (\partial U_z / \partial y)] )</td>
<td>(F)</td>
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<td>( \tau_{rr} = -2\mu(\partial U_r / \partial r) + (2\mu/3)(\nabla \cdot U) )</td>
<td>(G)</td>
</tr>
<tr>
<td>( \tau_{\theta\theta} = -2\mu\left[\frac{1}{r}\left( \frac{\partial U_\theta}{\partial \theta} \right) + \frac{U_r}{r} \right] + (2\mu/3)(\nabla \cdot U) )</td>
<td>(H)</td>
</tr>
<tr>
<td>( \tau_{zz} = -2\mu(\partial U_z / \partial z) + (2\mu/3)(\nabla \cdot U) )</td>
<td>(I)</td>
</tr>
<tr>
<td>( \tau_{r\theta} = \tau_{\theta r} = -\mu\left[ r \frac{\partial}{\partial r}(U_\theta/r) + \frac{1}{r} \left( \frac{\partial U_r}{\partial \theta} \right) \right] )</td>
<td>(J)</td>
</tr>
<tr>
<td>( \tau_{\theta z} = \tau_{z\theta} = -\mu\left[ (\partial U_\theta / \partial z) + \frac{1}{r} \left( \frac{\partial U_z}{\partial \theta} \right) \right] )</td>
<td>(K)</td>
</tr>
<tr>
<td>( \tau_{rz} = \tau_{zr} = -\mu[(\partial U_r / \partial z) + (\partial U_z / \partial r)] )</td>
<td>(L)</td>
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</thead>
<tbody>
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<td>(M)</td>
</tr>
<tr>
<td>( \tau_{\theta\theta} = -2\mu\left[\frac{1}{r}\left( \frac{\partial U_\theta}{\partial \theta} \right) + \frac{U_r}{r} \right] + (2\mu/3)(\nabla \cdot U) )</td>
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<td>( \tau_{\phi\phi} = -2\mu\left[ \frac{1}{r \sin \theta} \frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r} + (U_\theta/r)(\cot \theta) \right] + (2\mu/3)(\nabla \cdot U) )</td>
<td>(O)</td>
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<td>( \tau_{r\theta} = \tau_{\theta r} = -\mu\left[ r \frac{\partial}{\partial r}(U_\theta/r) + \frac{1}{r} \left( \frac{\partial U_r}{\partial \theta} \right) \right] )</td>
<td>(P)</td>
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<tr>
<td>( \tau_{r\phi} = \tau_{\phi r} = -\mu\left[ \frac{\sin \theta}{r} \left( \frac{\partial}{\partial \theta} \left( \frac{U_\phi}{\sin \theta} \right) \right) + \frac{1}{r \sin \theta} \frac{\partial U_\phi}{\partial \phi} \right] )</td>
<td>(Q)</td>
</tr>
<tr>
<td>( \tau_{\theta\phi} = \tau_{\phi \theta} = -\mu\left[ \frac{1}{r \sin \theta} \frac{\partial U_r}{\partial \phi} + r \frac{\partial}{\partial r}\left( \frac{U_\phi}{r} \right) \right] )</td>
<td>(R)</td>
</tr>
</tbody>
</table>

\(^2\) B&H *ibid* Table 5.2 p137.
Table 3: The energy equation\textsuperscript{3}

**General equation**

\[
\frac{\partial (\rho c_p T)}{\partial t} + (U \cdot \nabla) (\rho c_p T) = \dot{\bar{T}}_G + [\nabla \cdot \alpha \nabla (\rho c_p T)] - (\rho c_p T) (\nabla \cdot U)
\]

\[\text{(5.13)}\]

**Incompressible media, rectangular coordinates**

\[
\frac{\partial T}{\partial t} + U_x \frac{\partial T}{\partial x} + U_y \frac{\partial T}{\partial y} + U_z \frac{\partial T}{\partial z} = \frac{\dot{\bar{T}}_G}{\rho c_p} + \frac{\partial}{\partial x} \left( \alpha \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \alpha \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \alpha \frac{\partial T}{\partial z} \right)
\]

\[\text{(A)}\]

**Incompressible media, cylindrical coordinates**

\[
\frac{\partial T}{\partial t} + U_r \frac{\partial T}{\partial r} + \frac{U_\theta}{r} \frac{\partial T}{\partial \theta} + U_z \frac{\partial T}{\partial z} = \frac{\dot{\bar{T}}_G}{\rho c_p} + \frac{1}{r} \frac{\partial}{\partial r} \left( \alpha \frac{\partial T}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \alpha \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \alpha \frac{\partial T}{\partial z} \right)
\]

\[\text{(B)}\]

**Incompressible media, spherical coordinates**

\[
\frac{\partial T}{\partial t} + U_r \frac{\partial T}{\partial r} + \frac{U_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{U_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} = \frac{\dot{\bar{T}}_G}{\rho c_p} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \alpha \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \alpha \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( \alpha \frac{\partial T}{\partial \phi} \right)
\]

\[\text{(C)}\]

Table 4: The continuity equation for species \textit{A}\textsuperscript{4}

**General equation**

\[
\frac{\partial C_A}{\partial t} + (U \cdot \nabla) C_A = \dot{\bar{C}}_{A,G} + (\nabla \cdot D \nabla C_A) - (C_A) (\nabla \cdot U)
\]

\[\text{(5.8)}\]

**Incompressible media, rectangular coordinates**

\[
\frac{\partial C_A}{\partial t} + U_x \frac{\partial C_A}{\partial x} + U_y \frac{\partial C_A}{\partial y} + U_z \frac{\partial C_A}{\partial z} = \dot{\bar{C}}_{A,G} + \frac{\partial}{\partial x} \left( D \frac{\partial C_A}{\partial x} \right) + \frac{\partial}{\partial y} \left( D \frac{\partial C_A}{\partial y} \right) + \frac{\partial}{\partial z} \left( D \frac{\partial C_A}{\partial z} \right)
\]

\[\text{(A)}\]

**Incompressible media, cylindrical coordinates**

\[
\frac{\partial C_A}{\partial t} + U_r \frac{\partial C_A}{\partial r} + \frac{U_\theta}{r} \frac{\partial C_A}{\partial \theta} + U_z \frac{\partial C_A}{\partial z} = \dot{\bar{C}}_{A,G} + \frac{1}{r} \frac{\partial}{\partial r} \left( D \frac{\partial C_A}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( D \frac{\partial C_A}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( D \frac{\partial C_A}{\partial z} \right)
\]

\[\text{(B)}\]

**Incompressible media, spherical coordinates**

\[
\frac{\partial C_A}{\partial t} + U_r \frac{\partial C_A}{\partial r} + \frac{U_\theta}{r} \frac{\partial C_A}{\partial \theta} + \frac{U_\phi}{r \sin \theta} \frac{\partial C_A}{\partial \phi} = \dot{\bar{C}}_{A,G} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( D \frac{\partial C_A}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( D \frac{\partial C_A}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( D \frac{\partial C_A}{\partial \phi} \right)
\]

\[\text{(C)}\]

\textsuperscript{3} B\&H \textit{ibid} Table 5.6 p143.

\textsuperscript{4} B\&H \textit{ibid} Table 5.4 p142.