National Examination – Dec 2015
04-BS-16: Discrete Mathematics
Duration: 3 hours

Examination Type: Closed Book.
No aids allowed.

This exam paper contains 13 pages (including this one).
Answer 10 out of 12 questions. Ten questions constitute a full paper.
Please clearly indicate which two questions you don’t want marked by
drawing a diagonal line across the page.
In case of doubt to any question, clearly state any assumptions made.
One of two calculators is permitted any Casio or Sharp approved models.

# 1: ____ / 10
# 2: ____ / 10
# 3: ____ / 10
# 4: ____ / 10
# 5: ____ / 10
# 6: ____ / 10
# 7: ____ / 10
# 8: ____ / 10
# 9: ____ / 10
# 10: ____ / 10
# 11: ____ / 10
# 12: ____ / 10

TOTAL: ____ / 100

Good Luck!
Question 1. [10 marks]

Part (a) [2 marks]
Rewrite the following without negation on qualifiers: \( \neg \exists x \forall y \neg \exists z P(x, y, z) \)

Part (b) [2 marks]
Write the sentence “A necessary condition for \( P(x, y) \) to be true is that \( x > y \)” as a logic expression.

Part (c) [3 marks]
Is \( \exists x \forall y P(x, y) \rightarrow \forall x \exists y P(x, y) \) a tautology? Please either provide a proof or give a counterexample.

Part (d) [3 marks]
Consider the universe of discourse as positive integers. Let \( P_n(x, y, z) \) stand for \( x^n + y^n = z^n \). Write the Fermat’s Last Theorem as a logical proposition, i.e., the equation \( x^n + y^n = z^n \) does not have positive integer solution for \( n > 2 \).
**Question 2.** [10 MARKS]

**Part (a) [5 MARKS]**

Show that

\[ \sum_{s_1=0}^{1} \sum_{s_2=0}^{1} \cdots \sum_{s_n=0}^{1} \frac{1}{1^{s_1} 2^{s_2} \cdots n^{s_n}} = n + 1 \]

**Part (b) [5 MARKS]**

Show that the sum of even numbers from 0, 2, \cdots to 2n is \( n(n + 1) \).
Question 3. [10 marks]

Consider a sequence recursively defined as follows: $a_0 = 2$, $a_{n+1} = a_n^2$.

Part (a) [2 marks]

Write down a closed-form expression for $a_n$.

Part (b) [3 marks]

Is $a_n = O(2^n)$? Is $a_n = O(n^n)$?

Part (c) [5 marks]

Prove that $a_n - 1$ has at least $n$ distinct prime divisors.
Question 4. [10 marks]
A 5-card poker hand is dealt from a 52-card deck. Find the probability of getting

a. Five cards of consecutive rank (2 is the smallest rank, A largest).

b. There is at least one card of each suite.

c. All five cards come from the same suite.

d. There is exactly one pair.

e. Full house: three cards of same rank, plus a pair of different rank.
Question 5. [10 marks]

In the world series, two teams play a sequence of up to 7 games. The first team that wins 4 games wins the series. Assume that the teams are evenly matched.

Part (a) [2 marks]

What is the probability that the series ends after 4 games?

Part (b) [3 marks]

What is the probability that the series ends after the 5th game?

Part (c) [3 marks]

What is the probability that the series ends after the 6th game?

Part (d) [2 marks]

What is the probability that the series goes to the 7th game?
Question 6. [10 marks]

Part (a) [6 marks]

Suppose that we have 6 men and 4 women. How many different ways that

a. They can sit in a circular table so that all women sit next to each other? (clockwise and counter-clockwise seatings are regarded as different)

b. A committee of 5 people can be formed so that at most one of John, Mary and Susan is on the committee?

c. A committee of 5 people can be formed with more women than men?

Part (b) [4 marks]

How many ways there are to re-arrange the letters in SCIENCE, if

a. there are no restrictions?

b. the C's are together
Question 7. [10 marks]

Part (a) [4 marks]
Consider the relation $R$ defined on real numbers, where $(a, b) \in R$ if and only if $a - b$ is an integer. Show that $R$ is an equivalence relation. Describe the equivalence classes.

Part (b) [6 marks]
Plot the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin(x) + x$ over $x \in [-10, 10]$. Is this function one-to-one? onto? Does it have an inverse? If not, specify the largest sets $\mathcal{X}$ and $\mathcal{Y}$ for which the function $f : \mathcal{X} \to \mathcal{Y}$ has an inverse.
Question 8. [10 marks]

Part (a) [6 marks]

Show that a Fibonacci sequence with the initial condition $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ can be written in closed-form as

$$a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

Part (b) [4 marks]

Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$
Question 9. [10 marks]

Part (a) [2 marks]
Provide a definition of what it means by $f(n)$ is $O(g(n))$?

Part (b) [4 marks]
Insertion sort builds a sorted list by inserting one item to the list at a time. Describe how the algorithm works. What is the best-case, the worst-case, and the average run-time complexity of insertion sort? Please explain and provide adequate justification.

Part (c) [1 mark]
Write down the name of a sorting algorithm that has better average run-time complexity than insertion sort.

Part (d) [3 marks]
Please order the following run-time complexity in big-$O$ notation from slowest to fastest.

$O(n^2), O(n\sqrt{n}), O(\log(n)), O((\log(n))^2), O(\log(\log(n))), O(2^n), O(n^2 \log(n)), O(1)$
Question 10. [10 marks]

Part (a) [2 marks]
Let $G$ be a connected planar simple graph with $e$ edges, and $v$ vertices. Let $f$ be the number of regions in the planar representation of $G$ (including the outer region). What is the relation between $e$, $f$ and $v$?

Part (b) [2 marks]
A truncated tetrahedron has 4 hexagonal faces and 4 triangle faces. How many vertices and how many edges does it have?

Part (c) [3 marks]
Suppose that you use 20 equilateral triangles of same size as faces to construct a polyhedron, you will get a regular icosahedron. How many triangles meet around each vertex?

Part (d) [3 marks]
A truncated rhombic dodecahedron consists of square faces and hexagon faces. It has 48 edges and 32 vertices. How many faces are squares and how many hexagons?
**Question 11.** [10 marks]

**Part (a) [2 marks]**

What is an Euler circuit of a graph? Under what condition does a graph have a Euler circuit?

**Part (b) [3 marks]**

For what values of \((m, n)\) does \(K_{m,n}\), the complete bipartite graph with \(m\) vertices on one side and \(n\) vertices on the other, have a Euler circuit? Explain.

**Part (c) [2 marks]**

What is a Hamilton path of a graph?

**Part (d) [3 marks]**

Illustrate whether tetrahedron (four triangle faces), cube (six square faces), and octahedron (eight triangle faces) have a Hamilton path or not.
Question 12. [10 marks]

In some cultures, families prefer boys to girls. Suppose that in a society all families keep having more children until a boy is born (and they stop having children as soon as a boy is born). Assume that boys and girls are born with equal probability.

Part (a) [3 marks]

Give an expression for the average number of children per family in this society.

Part (b) [2 marks]

Give an expression for the average number of girls per family in this society.

Part (c) [1 mark]

What is the average number of boys per family in this society?

Part (d) [3 marks]

Would this society have an imbalance between males and females in the population over the long run? Please explain why or why not.

Total Marks = 100